ON THE MEAN LENGTH OF THE CHORDS OF A CLOSED CURVE

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ABSTRACT

Let us consider the N-gons with unit length of sides in the plane. What is the maximum of the arithmetical mean of the length of diagonals? We give an elementary solution for this problem and some more general ones. We deal with continuous analogons too.

1. W. Blaschke [1] has considered the integral

$$I = \frac{1}{A^2}\int_D\int_D f(r_{PQ})dt_P dt_Q, \quad A \text{ is fixed,}$$

where P, Q are points of the bounded convex domain D in the plane, r_{PQ} is the distance between P and Q, tp and t_Q are area-elements of D, $f(x)$ is a function such that I exists and $f'(x) < 0, f''(x) < 0$, and A is the area of D. He proved that I attains its maximum only at circles. His main device is the well-known Steiner's symmetrising procedure.

Making some extension of this procedure, T. Carleman has weakened the conditions of convexity of D and concavity of $f(x)$.

In this paper we shall deal with an analogous problem raised by I. Vincze,* namely the maximalization of the expression

$$\mathscr{F} = \frac{1}{L^2}\int_C\int_C g(r_{PQ})ds_P ds_Q, \quad L \text{ is fixed,}$$

where P, Q are points of a closed curve C, r_{PQ} is the distance between them, s_P, s_Q are arc-length parameters, L is the length of C and $g(t)$ is a function about which we make some assumptions. Our result is contained in

THEOREM I. *If the function $g(t)$ is increasing and concave, the integral $\mathscr{F}$ attains its maximum only at circles. In particular, taking $g(t) = t$, the mean length of chords of a closed curve C does not exceed the value $2L/\pi^2$.*

We now mention some further results used in the proof of Theorem 1. In this direction see [3].

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* Many thanks to him for his many suggestions to simplify the proofs.

We define a "system $(A_i)_N$" to be a sequence of points $A_1, A_2, \cdots, A_N$ and $A_{N+i} = A_i (i = 1, 2, \cdots, N)$. Such a system will be called regular, which will be denoted by (R_i), if the polygon $A_1, A_2, \cdots, A_N$ is regular. In this case, the following statements are valid.

THEOREM II. *Making use of the notation* $r_{il} = \overline{A_i A_{i+l}}$, *if* $r_{i1} \leqq a, (i = 1, \cdots, N)$ *and* $g(t)$ *is an increasing, concave function, the inequality*

$$(1) \qquad \frac{1}{N} \sum_{i=1}^{N} g(r_{il}^2) \leqq g\left(a^2 \sin^2 \frac{l\pi}{N} \Big/ \sin^2 \frac{\pi}{N}\right) \qquad l \text{ given}, \ N \geqq 4,$$

is valid. The sign of equality holds if and only if $(A_i)_N = (R_i)_N$.

THEOREM III. *For an arbitrary system* $(A_i)_N$ *the following relation is valid:*

$$(2) \qquad \sum_{i=1}^{N} r_{il}^2 \leqq \left(\sin \frac{l\pi}{N} \Big/ \sin \frac{\pi}{N}\right)^2 \sum_{i=1}^{N} r_{i_1}^2 \qquad l \text{ given}, \ N \geqq 4,$$

where equality holds only for affine images of a regular system and their limit cases.

2. To prove our theorems we need some notations and lemmas.

Let be T the set of the affine transformations of our plane σ. Then the lemmas below hold:

LEMMA 1. *Let us assume, that* A, B, C, *are not collinear and the same is true for* A', B', C'. *Then there exists one and only one* $\tau \in T$ *for which*

$$\tau(A) = A'$$
$$\tau(B) = B'$$
$$\tau(C) = C'$$

hold.

LEMMA 2. *The parallelism of straight lines and the ratio of diversion on a straight line will remain unchanged after applying a* $\tau \in T$.

LEMMA 3. *If segments* AB *and* CD *are parallel, then*

$$\overline{\tau(A)\tau(B)} : \overline{\tau(C)\tau(D)} = \overline{AB} : \overline{CD}$$

LEMMA 4. *A non-collinear system* $(A_i)_N$ *can be mapped into a regular one* $(R_i)_N$ *if and only if*

a) $R_i R_j$ *parallel to* $R_s R_t$ *implies that* $A_i A_j$ *is parallel to* $A_s A_t$.

b) $\overline{A_i A_j} : \overline{A_s A_t} = \overline{R_i R_j} : \overline{R_s R_t}$

if $A_i A_j$ *and* $A_s A_t$ *are parallel.*

Proof. Only part of our lemma follows directly from Lemmas 3 and 2. But if a) and b) are satisfied by $(A_i)_N$, consider $\tau \in T$ defined by $\tau(R_i) = A_i$, $i = 1, 2, 3$

(see Lemma 1). (Obviously A_1, A_2, A_3 cannot be collinear because in this case $\{A_i\}_N$ would be collinear.)

Let us denote $\tau(R_i)$ by R_i' $(i = 1, 2, \cdots, N)$, from condition a) and Lemma 2, A_4 and R_4 must lie on the straight line parallel with A_2A_3 and containing A_1. On the other hand

$$\overline{A_1R_4'} = \overline{A_1A_4} = \overline{A_2A_3}\ (\overline{R_1R_4}:\ \overline{R_2R_3})$$

So $R_4' = A_4$, etc.

LEMMA 5. *Let the numbers* x_i $(x_i \geqq 0)$ *satisfy the relations*

(3) $$x_j \leqq x_1 + (x_{j-1}x_{j+1})^{1/2} \qquad (j = 2, \cdots, N-2)$$
$$x_{N-i} = x_i$$

Then

$$x_j \leqq x_1 \left(\sin j\frac{\pi}{N} \Big/ \sin\frac{\pi}{N}\right)^2 \qquad (j = 2, \cdots, N-2)$$

and if we have

$$x_j < x_1 + (x_{i-1}x_{i+1})^{1/2} \quad \textit{for some } j \quad (1 < j < N-1)$$

then

$$x_j < x_1 \left(\sin j\frac{\pi}{N} \Big/ \sin\frac{\pi}{N}\right)^2 \textit{ for every } j \quad (1 < j < N-1)$$

Proof. Let $y_j = x_j/x_1$ $(i = 2, \cdots, N-2)$. In view of $(ab)^{1/2} \leqq (a + b)/2$ $(a, b > 0)$ system (3) implies that

$$y_2 \leqq 1 + \frac{1}{2}\frac{v_3}{v_1} + \frac{1}{2}\frac{v_1}{v_3}y_3$$

$$y_3 \leqq 1 + \frac{1}{2}\frac{v_4}{v_2}y_2 + \frac{1}{2}\frac{v_2}{v_4}y_4$$

$$y_4 \leqq 1 + \frac{1}{2}\frac{v_5}{v_3}y_3 + \frac{1}{2}\frac{v_3}{v_5}y_3$$

$$\vdots$$

(4) $$y_i \leqq 1 + \frac{1}{2}\frac{v_{i+1}}{v_{i-1}}y_{i-1} + \frac{1}{2}\frac{v_{i-1}}{v_{i+1}}y_{i+1}$$

$$\vdots$$

$$y_{k-1} \leqq 1 + \frac{1}{2}\frac{v_k}{v_{k-2}}y_{k-2} + \frac{1}{2}\frac{v_{k-2}}{v_k}y_k$$

$$y_k \leqq 1 + \frac{1}{2}\frac{v_k}{v_{k-1}}y_{k-1} + \frac{1}{2}\frac{v_{k-1}}{v_k}y_k \qquad (N = 2k + 1)$$

$$y_k \leqq 1 + y_{k+1} \qquad (N = 2k)$$

where $v_1, v_2 \cdots, v_k > 0$. Let $v_i = \sin i\,(\pi/N)$. Suppose that $N = 2k+1$ and write (4) in matrix form

$$Y \leqq AY + B. \tag{5}$$

As

$$\sum_{i=2}^{k} a_{ij} = \delta_j < 1, \qquad (j = 2, \cdots, k)$$

summing the relations of (5) we get

$$\sum_{i=2}^{k} c_j x_j \leqq M \qquad \left(c_i = 1 - \delta_i > 0,\ M = \sum_{j=2}^{k} b_j\right..$$

So every y_i is bounded:

$$y_i \leqq M/c_i.$$

If $y_j < (AY)_j + B_j$ for some j $(1 < j < k)$ then we can increase the value of y_j to y'_j so that

$$Y' = (y_2, y_3, \cdots, y_{j-1}; y'_j, y_{j+1}, \cdots, y_{k-1}, y_k) \in S = \,< U : U \leqq AU + B >.$$

Now we can increase the value of y_{j-1} because $y_{j-1} < [AY']_{j-1} + b_{j-1}$ if $j > 2$. This procedure can also be carried out for $x_{j-2}, \cdots, x_2$ and $x_{j+1}, \cdots, x_k$, if they exist. Therefore if

$$Y = \max_{U \in S} U$$

then

$$Y = AY + B \tag{6}$$

This equation has the solution

$$Y^0 = \,< \left(\sin \frac{\pi}{N} i \Big/ \sin \frac{\pi}{N}\right)^2 >_{i=2}^{k}$$

But this solution is unique. For let Y^1 be any other solution of (6). Let $W = Y^0 - Y^1$. Then $W = AW$ and so

$$\begin{aligned}
|w_2| &= a_{23}|w_3| \\
|w_3| &\leqq a_{32}|w_2| + a_{34}|w_4| \\
&\vdots \\
|w_i| &\leqq a_{ii-1}|w_{i-1}| + a_{ii+1}|w_{i+1}| \\
&\vdots \\
|w_{k-1}| &\leqq a_{k-1\,k-2}|w_{k-2}| + a_{k-1\;k+1}|w_k| \\
|w_k| &\leqq a_{k\,k-1}|w_{k-1}| + a_{k\;k+1}|w_k|
\end{aligned}$$

Summing these inequalities we get

$$\sum_{j=2}^{k} c_j |w_j| \leqq 0 \qquad (c_j = 1 - \delta_j > 0,\ j = 2, \cdots, k)$$

Hence $w_j = 0,\ j = 2, \cdots, k$.

In the case $N = 2K$ we can eliminate the variable y_k by replacing the last relation of (4) into the preceding one. The new system $Y \leqq A'Y + B'$ can be considered as (5) and the result is that

$$y_j \leqq \left(\sin \frac{\pi}{N} j \Big/ \sin \frac{\pi}{N}\right)^2,\ j = 2, \cdots, k-1.$$

But

$$y_k \leqq 1 + y_{k-1} \leqq 1 + \left(\sin \frac{\pi}{2k}(k-1) / \sin \frac{\pi}{2k}\right)^2 =$$

$$= \frac{\sin^2 \frac{\pi}{2k}(k-1) + \sin^2 \frac{\pi}{2k}}{\sin^2 \frac{\pi}{2k}} = \frac{1}{\sin^2 \frac{\pi}{2k}} = \frac{\sin^2 k\frac{\pi}{2k}}{\sin^2 \frac{\pi}{2k}},$$

which completes the proof of Theorem 1. (see Remark 1).

LEMMA 6. *Let the functions $f(t)$ and $g(t)$ be defined in $[0,c]\,(c>0)$ and let us assume that $f(t) = f(c-t)$, $g(t) = g(c-t)$ for every $0 \leqq t \leqq c$. Let us denote by $A\ (x,y,z)$ a function strictly increasing in x,y,z. If the relations*

$$f(u) \leqq A\{f(v), f(u-v), f(u+v)\}$$

$$g(u) = A\{g(v), g(u-v), g(u+v)\} \qquad 0 \leqq u-v, u+v \leqq c$$

$$f(t) \leqq g(t), \quad 0 \leqq t \leqq c$$

hold, then

$$f(t_0) = g(t_0) \quad t_0 \in (0,c)$$

implies that

$$f(t) \equiv g(t), \quad 0 \leqq t \leqq c.$$

Proof. We may assume that $0 < t_0 < c/2$. Let $0 < v < t_0$. Then

$$f(t_0) \leqq A\{f(v), g(t_0 - v), g(t_0 + v)\} \leqq g(t_0).$$

Because $f(t_0) = g(t_0)$ and as A is strictly increasing, we have

$$f(v) = g(v) \qquad 0 \leqq v \leqq t_0$$

$$f(t_0+v) = g(t_0+v)$$

i.e.

$$f(t) \equiv g(t), \ 0 \leqq t \leqq 2t_0.$$

If $t_0 \geqq c/4$, the lemma is proved. In case $t_0' = 2t_0 < c/2$ we come to the conclusion that $f(t) = g(t)$ in $[0, 2t_0']$, and so on.

LEMMA 7. *For an arbitrary quadrangle A, B, C, D, with sides of length a, b, c, d, and diagonals of length e, f,*

(6) $$e^2 + f^2 \leqq b^2 + d^2 + 2ac$$

is valid with equality only in the case of AB and CD parallel.

Proof. Let us denote the vectors AB, BC, CD, DA, by v_1, v_2, v_3, v_4. Then $v_1 + v_2 + v_3 + v_4 = 0$ and

$$\begin{aligned} e^2+f^2 &= \frac{1}{2}\{\|v_1+v_2\|^2 + \|v_2+v_3\|^2 + \|v_3+v_4\|^2 + \|v_4+v_1\|^2\} \\ &= \sum_{i=1}^{4} \|v_i\|^2 + (v_1,v_2) + (v_2,v_3) + (v_3,v_4) + (v_4,v_1) \\ &= \sum_{i=1}^{4} \|v_i\|^2 + (v_1+v_3, v_2+v_4) \\ &= \sum_{i=1}^{4} \|v_i\|^2 - \|v_1+v_3\|^2 \leqq \sum_{i=1}^{4} \|v_i\|^2 - (\|v_1\| - \|v_3\|)^2 \\ &= \|v_2\|^2 + \|v_4\|^2 + 2\|v_1\| \, \|v_3\| \\ &= b^2 + d^2 + 2ac \end{aligned}$$

3. Proof of Theorem 3. For an arbitrary system $\{A_i\}_N$ the relation

$$r_{il}^2 + r_{i+1,l}^2 \leqq r_{i,1}^2 + r_{i+l,1}^2 + 2r_{i,l+1}\, r_{i+1,l-1}$$

holds by Lemma 7. Hence

(7) $$\begin{aligned} \sum_{i=1}^{N} r_{il}^2 &\leqq \sum_{i=1}^{N} r_{i,1}^2 + \sum_{i=1}^{N} r_{i,l+1} r_{i+1,l-1} \\ &\leqq \sum_{i=1}^{N} r_{i,1}^2 + \left\{ \sum_{i=1}^{N} r_{i,l-1}^2 \sum_{i=1}^{N} r_{i,l+1}^2 \right\}^{1/2}. \qquad (l = 2, 3, \cdots, N-2) \end{aligned}$$

Let us write

$$\rho_l = \sum_{i=1}^{N} r_{il}^2 .$$

Then $\{\rho_i\}_1^{N-1}$ satisfy the system (3). From Lemma 5 it follows that

$$\rho_l \leqq \rho_1\left(\sin l\frac{\pi}{N} \Big/ \sin\frac{\pi}{N}\right)^2 = q_l$$

and if $\rho_l = q_l\rho_1$ for some l $(1 < l \leqq N-2)$ then the equations

$$\rho_l = \rho_1 + (\rho_{l-1}\rho_{l+1})^{1/2} \qquad (l = 2,\cdots,N-2)$$

hold. But in this case — because Cauchy's inequality was applied in (7) —

$$\frac{r_{i,l+1}}{r_{i+1,l-1}} = \left(\frac{\rho_{l+1}}{\rho_{l-1}}\right)^{1/2} = \frac{\sin(l+1)\frac{\pi}{N}}{\sin(l-1)\frac{\pi}{N}}$$

and the segments A_iA_{i+l+1} and $A_{i+1}A_{i+l}$ are parallel (see Lemma 7), if $\{A_i\}_N$ is a non-collinear system. By Lemma 4, $A_i = \tau(R_i)$ in this case.

Suppose that $\{A_i\}_N$ is maximal and collinear and

$$A_i = (x_i, 0) \qquad i = 1,\cdots,N$$

in a Cartesian coordinate system.*

Equations

$$x_{i+1} - x_i = \Delta(x_{i+2} - x_{i+1})$$

cannot hold if $\Delta \neq 0$, as $x_{N+i} = x_i$, $i=1,\cdots,N$, and we exclude the case $\Delta = 0$ since, in this case $x_1 = x_2 = \cdots = X_N$ would be satisfied. Therefore there exist such i and j that

$$(8) \qquad \begin{aligned} x_{i+1} - x_j &= \Delta_i(x_{i+2} - x_{i+1}) \\ x_{j+1} - x_1 &= \Delta_j(x_{j+2} - x_{i+1}) \end{aligned} \qquad \Delta_i \neq \Delta_j.$$

Taking into consideration the system $\{B_i\}_N = \{x_i, y_i\}$ where

$$y_i = x_{i+1}\ (i,= 1,\cdots,N), \{B_i\}_N$$

cannot be collinear because of (8). It is obvious that $\{B_i\}_N$ is maximal:

$$\frac{\sum\{(x_i - x_{i+l})^2 + (y_i - y_{i+1})^2\}}{\sum\{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2\}} = \frac{2\sum(x_i - x_{i+l})^2}{2\sum(x_i - x_{i+1})^2}.$$

We have $B_i = \tau(R_i)$, $A_i = P(B_i)$. So $A_i = P(\tau(R_i)) = P'(R_i)$, where P, P' are parallel projections into the line $y = 0$, and $\tau \in T$.

*We may assume that $\{A_i\}_N$ has more than two different points.

Proof of theorem II. It is evident from the theorem above and Jensen's inequality.

Proof of Theorem I. Let $0 < u < L$ and $\{l_N\}$ be a sequence of integers such that $l_N/N \to u$ $(N \to \infty)$.

Let us assume that the sequences of equilateral polygons $\{\pi_N\}, \{\pi'_N\}$ inscribed in a curve C and a circle K, both with length L, converges to C and K, respectively.

Applying Theorem II with $g(t) = t$ at each N, we get clearly $Na_N \to L$ and so

$$(9) \qquad \frac{1}{L}\int_{\widehat{PP'}=u} r^2_{PP'} ds_P \leqq \left(\frac{L}{\pi}\sin\frac{u}{L}\pi\right)^2$$

where u is given, and $\widehat{PP'}$ is the length of the arc of C between P and P' going from P to P' counter-clockwise.

The relation (9) implies that

$$\mathscr{F}(u) = \frac{1}{L}\int_{\widehat{PP'}=u} g(r^2_{PP'})ds_P \leqq g\left\{\left(\frac{L}{\pi}\sin\frac{u}{L}\pi\right)^2\right\}$$

(9 is a monotone increasing and concave function), and if equality holds here for some u and C, then equality holds for this same u and C in (9) and so

$$(10) \qquad \rho_c(u) \overset{def}{\equiv} \frac{1}{L}\int_{\widehat{PP'}=u} r^2_{PP'} ds_P \equiv \left(\frac{L}{\pi}\sin\frac{u}{L}\pi\right)^2 \overset{def}{\equiv} \rho_k(u),$$

as the conditions of Lemma 6 are satisfied with

$$f(t) = \rho_c(t),\ g(t) = \rho_k(t),\ A(x,y,z) = x + (yz)^{1/2}.$$

(The inequality $\rho(u) \leqq \rho(v) + [\rho(u+v)\rho(u-v)]^{1/2}$ can be derived from Lemma 7 as we had obtained the relation $\rho_i \leqq \rho_1 + (\rho_{i-1}\rho_{i+1})^{1/2}$ in the proof of Theorem III).But (10) implies the identity

$$(11) \qquad \phi_c(u,v) \overset{def}{\equiv} -\rho_c(u) + \rho_c(v) + (\rho_c(u+v)\rho_c(u-v))^{1/2} \equiv 0.$$

Hence

$$(12) \qquad \psi_c(s_P,u,v) \overset{def}{\equiv} r^2_{PP_1} + r^2_{P_2P_3} + 2r_{P_1P_2}r_{PP_3} - r^2_{PP_2} - r^2_{P_1P_3} \equiv 0,$$

and

$$(13) \qquad \lambda_c(s_P,u,v) \overset{def}{\equiv} \frac{r_{P_1P_2}}{r_{PP_3}} \equiv \frac{\sin\dfrac{u+v}{L}\pi}{\sin\dfrac{u-v}{L}\pi}$$

$(P \in C;\ C,u,v$ given)

where P_1, P_2, P_3 are such that

$$\widehat{PP_2} = \widehat{P_1P_3} = u;\ \widehat{PP_1} = \widehat{P_2P_3} = v.$$

For $\psi_c(s_P, u, v) \geqq 0$;

$$\int_{P \in c} \psi_c(s_P, u, v)\, ds_P \leqq 2\phi_c(u, v) = 0, \tag{14}$$

$\psi_c(s_P, u, v)$ is continuous in s_P, and Cauchy's inequality has been applied in (14).

Let points $A_1, A_2, A_3, A_4, A_5 \in C$ be such that

$$\widehat{A_1A_2} = \widehat{A_2A_3} = \widehat{A_3A_4} = \widehat{A_4A_5} = \widehat{A_5A_1} = L/5. \tag{15}$$

Now we construct a sequence π_N of polygons inscribed in C such that $\pi_N = \tau(\pi_N')$ where $\{\pi_N'\} \subset K$ is regular, and $\tau \in T$.

Let π_1 consist of the points A_i $(i = 1, 2, \cdots, 5)$. If we have $\pi_N = < A_i^N, i = 1, \cdots, 5^N >$ then the points of π_{N+1} are defined by the relations

$$\widehat{A_i^{N+1} A_{i+1}^{N+1}} = \frac{L}{5^{N+1}},\ i = 1, \cdots, 5^N,\ \pi_N \subseteq \pi_{N+1}.$$

Suppose that C is not collinear. In this case π_N has the same property if N is sufficiently large: $N \geqq N_0$.

From (12), (13) and Lemma 4

$$\pi_N = \tau_N(\pi_N') \qquad (N = N_0, N_{0+1}, \cdots)$$

But evidently

$$\tau_{N_0} = \tau_{N_0+1} = \tau_{N_0+2} = \cdots \overset{def}{=} \tau.$$

Letting N approach infinity, we get $C = \tau(K)$ i.e. C is an ellipse. The curve C cannot differ from K as in this case the affine mapping τ would change the length of K, which contradicts (15).

If points of C are collinear, then π_N is also collinear. $(N = 1, 2, 3, \cdots)$. Therefore from (12), and (13)

$$\pi_N = P_N(R_N)\ (N = 1, 2, \cdots),$$

P_N is a projection on the line containing π_N.

Evidently we have $P_1 = P_2 = \cdots = P_N = \cdots \overset{def}{=} P$, so $C = P(K)$. But this contradicts (15) by the above argument. Hence C cannot be collinear.

As we have

$$\mathscr{F} = \frac{1}{L}\int_0^1 \mathscr{F}(u)\, du,$$

our theorem is proved.

REMARK. 1) Let us assume that $Y \leqq AY + B$ with the notations of lemma 5.

It is easy to show, that the iteration $Y_0 = Y$, $Y_K = A\, Y_{K-1} + B$, $k = 1,2,\cdots$ is increasing and converges. This fact does more elementary the proof of this lemma.

2) Let be d the transfinite diameter of C. Then we have

$$d = \exp\left(\max_{\mu} \frac{1}{Q^2} \int_C \int_C \mu_P \mu_Q \log r_{PQ} ds_P ds_Q\right)$$

where $Q = \int_C \mu_P ds_P$ [5].

To examine the minimum of the integral

$$\frac{1}{L^2} \int_C \int_C \log r_{PQ}\, ds_P ds_Q$$

for convex curves is perhaps easier as as to examine the minimum of d, and this way we could get a good lower bound for d/L.

3) The classical isoperimetric inequality follows from Theorem 1 using the formula of area

$$16A^2 = \int_C \int_C r_{PQ}^2 \cos\alpha_{PQ}\, ds_P ds_Q$$

due to L. Rédei and B. Sz. Nagy [4].

4) Inequality of Wirtinger can be roved completely by a slight modification of our proof of Theorem 1.

LITERATURE

1. W. Blaschke, *Eine isoperimetrische Eigenschaft des Kreises*, Math. Zeitschrift I (1918), 52–58.

2. T. Carleman, *Über eine isoperimetrische Aufgabe und ihre physikalischen Anwendungen*, Math. Zeitschrift **3** (1919).

3. Fan, K. O. Taussky and J. Todd, *Discrete analogues of inequalities of Wirtinger*, Monatsh. Math. Physik **59** (1955), 73–90.

4. L. Rédei and B. Sz. Nagy, *Eine Vorallgemeinung der Heronische Formel*, Publ. Math. Debrecen (1950).

5. G. Pólya and G. Szegö, *Isoperimetric inequalities in Mathematical Physics*, Princeton University Press, (1959).

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